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A General Form of Alzer's Inequality

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Abstract—Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing positive sequence, and let m be a natural number and r a positive number. In this paper, we prove

(i) if

$$\left(\frac{a_n}{a_{n-1}}\right)^{(n-1)/n} \le \frac{a_{n+1}}{a_n} \le \frac{a_n}{a_{n-1}},$$

for $n \ge 2$, then

$$\frac{a_n}{a_{n+m}} < \left(\frac{(1/n)\sum_{i=1}^{n} a_i^r}{(1/(n+m))\sum_{i=1}^{n+m} a_i^r}\right)^{1/r},$$

for $n \ge 1$;

(ii) if

$$\frac{a_n}{a_{n+1}} \le \frac{\left(\prod_{i=1}^{n-1} a_i\right)^{1/(n-1)}}{\left(\prod_{i=1}^{n} a_i\right)^{1/n}},$$

for $n \ge 2$, then

$$\left(\frac{(1/n)\sum_{i=1}^{n} a_i^r}{(1/(n+m))\sum_{i=1}^{n+m} a_i^r}\right)^{1/r} < \frac{\left(\prod_{i=1}^{n} a_i\right)^{1/n}}{\left(\prod_{i=1}^{n+m} a_i\right)^{1/(n+m)}},$$

for $n \ge 1$.

An open problem proposed in [1], which concerns the sequence of natural numbers and might be a generalization of Alzer's inequality, is shown to be a special case of our second result. Relative results are also given. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The following inequality was proved in [1].

THEOREM 1.1. *Let n and m be natural numbers, k a nonnegative integer. Then, for any positive real number r, we have*

$$\frac{n+k}{n+m+k} < \left(\frac{(1/n) \sum_{i=1}^{n} (i+k)^r}{(1/(n+m)) \sum_{i=1}^{n+m} (i+k)^r} \right)^{1/r}. \tag{1}$$

Also in [1], the author posed a conjecture as follows.

CONJECTURE. *Let n and m be natural numbers, k a nonnegative integer. Then, for any positive real number r, we have*

$$\left(\frac{(1/n) \sum_{i=1}^{n} (i+k)^r}{(1/(n+m)) \sum_{i=1}^{n+m} (i+k)^r} \right)^{1/r} < \frac{\left(\prod_{i=1}^{n} (i+k) \right)^{1/n}}{\left(\prod_{i=1}^{n+m} (i+k) \right)^{1/(n+m)}}. \tag{2}$$

Both (1) and (2) are generalizations of the following Alzer's inequalities in [2]:

$$\frac{n}{n+1} \le \left(\frac{(1/n) \sum_{i=1}^{n} i^r}{(1/(n+1)) \sum_{i=1}^{n+1} i^r} \right)^{1/r} \le \frac{n!^{1/n}}{(n+1)!^{1/(n+1)}}. \tag{3}$$

Throughout this paper, let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing positive sequence, and let r be a positive number. Denote

$$A_n^{(r)} = \frac{1}{n} \sum_{i=1}^{n} a_i^r, \qquad G_n = \left(\prod_{i=1}^{n} a_i \right)^{1/n}.$$

It is well known that both $\{A_n^{(r)}\}_{n=1}^{\infty}$ and $\{G_n\}_{n=1}^{\infty}$ are strictly increasing positive sequences, and $a_n > A_n > G_n$ for $n \ge 2$. We are going to present a general form of Alzer's inequality (Theorem 1.2). The first part is a generalization of (1); from the second part it follows that (2) holds true for any natural numbers n and m, any nonnegative real number k, and any positive real number r, under a few additional arguments.

THEOREM 1.2. *Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing positive sequence, and let m be a natural number and r be a positive number. Then,*

(i) *if*

$$\left(\frac{a_n}{a_{n-1}} \right)^{(n-1)/n} \le \frac{a_{n+1}}{a_n} \le \frac{a_n}{a_{n-1}}, \qquad \text{for } n \ge 2, \tag{4}$$

then

$$\frac{a_n}{a_{n+m}} < \left(\frac{A_n^{(r)}}{A_{n+m}^{(r)}} \right)^{1/r}, \qquad \text{for } n \ge 1; \tag{5}$$

(ii) *if*

$$\frac{a_n}{a_{n+1}} \le \frac{G_{n-1}}{G_n}, \qquad \text{for } n \ge 2, \tag{6}$$

then

$$\left(\frac{A_n^{(r)}}{A_{n+m}^{(r)}} \right)^{1/r} < \frac{G_n}{G_{n+m}}, \qquad \text{for } n \ge 1. \tag{7}$$

Moreover, the lower bound in (5) and the upper bound in (7) are both best possible.

2. PROOF OF THEOREM 1.2

LEMMA 2.1. *Let $\{B_n\}_{n=1}^{\infty}$ and $\{C_n\}_{n=1}^{\infty}$ be strictly increasing positive sequences with*

$$\frac{B_1}{B_2} < \frac{C_1}{C_2}. \tag{8}$$

If

$$\frac{B_{n+1} - B_n}{B_{n+2} - B_{n+1}} \le \frac{C_{n+1} - C_n}{C_{n+2} - C_{n+1}}, \qquad \text{for } n \ge 1, \tag{9}$$

then

$$\frac{B_n}{B_{n+1}} < \frac{C_n}{C_{n+1}}, \qquad \text{for } n \ge 1. \tag{10}$$

PROOF. Denote

$$\Delta B_n = B_{n+1} - B_n, \qquad \Delta C_n = C_{n+1} - C_n.$$

Then, (9) can be rewritten as

$$\frac{\Delta B_n}{\Delta C_n} \le \frac{\Delta B_{n+1}}{\Delta C_{n+1}}. \tag{11}$$

And, (10) is equivalent to the following inequality:

$$\frac{B_n}{C_n} < \frac{(B_n + \Delta B_n)}{(C_n + \Delta C_n)},$$

which, in turn, is equivalent to the following inequality:

$$B_n < C_n \cdot \frac{\Delta B_n}{\Delta C_n}. \tag{12}$$

Now, we proceed by induction. Clearly, (10) or (12) holds for $n = 1$. Suppose that (12) holds for some $n \ge 1$. Then,

$$B_n + \Delta B_n < C_n \cdot \frac{\Delta B_n}{\Delta C_n} + \Delta B_n = (C_n + \Delta C_n) \cdot \frac{\Delta B_n}{\Delta C_n},$$

and, by (11) and the definitions of ΔB_n and ΔC_n,

$$B_{n+1} < C_{n+1} \cdot \frac{\Delta B_n}{\Delta C_n} \le C_{n+1} \cdot \frac{\Delta B_{n+1}}{\Delta C_{n+1}},$$

which means that (12) holds true for $n + 1$. The proof is complete. ∎

A series of auxiliary functions is needed in the proofs of the following two lemmas. For each n, define

$$g_n(x) = G_n^{n/(n+1)} x^{1/(n+1)}. \tag{13}$$

Then,

$$g_n(G_n) = G_n, \tag{14}$$

$$g_n(a_{n+1}) = G_{n+1}, \tag{15}$$

$$g_n'(x) = \frac{1}{n+1} G_n^{n/(n+1)} x^{1/(n+1)-1}, \tag{16}$$

$$g_n'(G_n) = \frac{1}{n+1}. \tag{17}$$

LEMMA 2.2. *For $n \geq 1$, we have*

$$G_{n+1} - G_n < \frac{1}{n+1}(a_{n+1} - G_n).$$

PROOF. Applying Lagrange's mean-value theorem to the function $g_n(x)$, and noticing (14) and (15), there is a $\xi \in (G_n, a_{n+1})$ such that

$$G_{n+1} - G_n = g_n'(\xi)(a_{n+1} - G_n).$$

Since $g_n'(x)$ is strictly decreasing on the interval (G_n, a_{n+1}), we obtain from the last equality that

$$G_{n+1} - G_n < g_n'(G_n)(a_{n+1} - G_n) = \frac{1}{n+1}(a_{n+1} - G_n).$$

The proof is complete. ∎

LEMMA 2.3. *Under condition (6), we have*

$$\frac{G_n - G_{n-1}}{G_{n+1} - G_n} > \frac{n+1}{n} \cdot \frac{a_n - G_{n-1}}{a_{n+1} - G_n}, \qquad \text{for } n \geq 2.$$

PROOF. For each n, define

$$f_n(x) = g_{n-1}\left(\frac{a_n - G_{n-1}}{a_{n+1} - G_n}(x - G_n) + G_{n-1}\right).$$

Then, similarly to (14)–(17), one has

$$f_n(G_n) = g_{n-1}(G_{n-1}) = G_{n-1}, \tag{18}$$

$$f_n(a_{n+1}) = g_{n-1}(a_n) = G_n, \tag{19}$$

$$f_n'(x) = \frac{1}{n} G_{n-1}^{(n-1)/n}\left[\frac{a_n - G_{n-1}}{a_{n+1} - G_n}(x - G_n) + G_{n-1}\right]^{1/n-1} \cdot \frac{a_n - G_{n-1}}{a_{n+1} - G_n}, \tag{20}$$

$$f_n'(G_n) = \frac{1}{n} \cdot \frac{a_n - G_{n-1}}{a_{n+1} - G_n}. \tag{21}$$

From Cauchy's mean-value theorem, noticing (14),(15),(18), and (19), using (16) and (20), one can compute easily that there is $\xi \in (G_n, a_{n+1})$, such that

$$\frac{G_n - G_{n-1}}{G_{n+1} - G_n} = \frac{f_n'(\xi)}{g_n'(\xi)} = Q(n)\alpha(\xi)\beta(\xi), \tag{22}$$

where

$$Q(n) = \frac{n+1}{n} \cdot \frac{G_{n-1}^{(n-1)/n}}{G_n^{n/(n+1)}} \cdot \frac{a_n - G_{n-1}}{a_{n+1} - G_n},$$

$$\alpha(\xi) = \left[\frac{a_n - G_{n-1}}{a_{n+1} - G_n}\xi + \left(G_{n-1} - \frac{a_n - G_{n-1}}{a_{n+1} - G_n}G_n\right)\right]^{1/(n(n+1))},$$

$$\beta(\xi) = \left[\frac{a_n - G_{n-1}}{a_{n+1} - G_n} + \frac{1}{\xi} \cdot \left(G_{n-1} - \frac{a_n - G_{n-1}}{a_{n+1} - G_n}G_n\right)\right]^{1/(n+1)-1}.$$

Notice that condition (6) is equivalent to

$$G_{n-1} - \frac{a_n - G_{n-1}}{a_{n+1} - G_n}G_n \geq 0,$$

and the fact that $(a_n - G_{n-1})/(a_{n+1} - G_n) > 0$, for each n, one always has

$$\alpha(\xi) > \alpha(G_n) > 0, \qquad \beta(\xi) \geq \beta(G_n) > 0. \tag{23}$$

The combination of (22) and (23) gives the desired result, using (17) and (21),

$$\frac{G_n - G_{n-1}}{G_{n+1} - G_n} > \frac{f_n'(G_n)}{g_n'(G_n)} = \frac{n+1}{n} \cdot \frac{a_n - G_{n-1}}{a_{n+1} - G_n}.$$

The proof is complete. ∎

LEMMA 2.4. *Let a, b, c, d, c', and d' be positive numbers satisfying*

$$c' > c, \qquad \frac{c'}{d'} < \frac{c}{d} < \frac{a}{b}. \tag{24}$$

Then

$$\frac{c+a}{d+b} > \frac{c'+a}{d'+b}. \tag{25}$$

PROOF. Denote

$$c = s_1 a, \quad d = s_2 b, \quad c' = t_1 c, \quad d' = t_2 d. \tag{26}$$

Substituting (26) into (25), we get an equivalent form of (25),

$$\frac{s_1+1}{s_2+1} > \frac{t_1 s_1 + 1}{t_2 s_2 + 1}. \tag{27}$$

Also, one can deduce from (24) and (26) easily that

$$s_2 > s_1 > 0, \qquad t_2 > t_1 > 1.$$

And then, one has both

$$(t_2 - 1)s_2 > (t_1 - 1)s_1, \tag{28}$$

and

$$1 + t_2 s_1 s_2 > 1 + t_1 s_1 s_2. \tag{29}$$

Adding (28) to (29), and rearranging, one will get (27). ∎

Now we are ready to prove Theorem 1.2.

PROOF OF THEOREM 1.2.

(i) We first prove the case $m = 1$; i.e.,

$$\frac{a_n}{a_{n+1}} < \left(\frac{A_n^{(r)}}{A_{n+1}^{(r)}}\right)^{1/r}, \qquad \text{for } n \geq 1. \tag{30}$$

Write (30) as

$$\frac{n a_n^r}{(n+1)a_{n+1}^r} < \frac{n A_n^{(r)}}{(n+1)A_{n+1}^{(r)}}, \qquad \text{for } n \geq 1. \tag{31}$$

It is clear that (30) or (31) holds for $n = 1$. Now we prove (31) for $n \geq 2$.

By Lemma 2.1, viewing B_n as $n a_n^r$ and C_n as $n A_n^{(r)}$, it suffices to show that

$$\frac{n a_n^r - (n-1)a_{n-1}^r}{(n+1)a_{n+1}^r - n a_n^r} \leq \frac{n A_n^{(r)} - (n-1)A_{n-1}^{(r)}}{(n+1)A_{n+1}^{(r)} - n A_n^{(r)}}, \qquad \text{for } n \geq 2,$$

or equivalently,

$$\frac{n a_n^r - (n-1)a_{n-1}^r}{(n+1)a_{n+1}^r - n a_n^r} \leq \frac{a_n^r}{a_{n+1}^r}, \qquad \text{for } n \geq 2. \tag{32}$$

Multiplying both sides of (32) by $a_{n+1}^r((n+1)a_{n+1}^r - n a_n^r)$, simplifying, and rearranging, one gets the equivalent form

$$\left(\frac{a_n}{a_{n+1}}\right)^r \leq \frac{1}{n} + \frac{n-1}{n}\left(\frac{a_{n-1}}{a_n}\right)^r, \qquad \text{for } n \geq 2. \tag{33}$$

For fixed $n \geq 2$, denote by $e_n(r)$ the difference of the two sides of (33),

$$e_n(r) = \left(\frac{a_n}{a_{n+1}}\right)^r - \frac{1}{n} - \frac{n-1}{n}\left(\frac{a_{n-1}}{a_n}\right)^r.$$

Then, $e_n(0) = 0$ and

$$n \cdot \frac{de_n(r)}{dr} = \left(\frac{a_n}{a_{n+1}}\right)^r \ln\left(\frac{a_n}{a_{n+1}}\right)^n - \left(\frac{a_{n-1}}{a_n}\right)^r \ln\left(\frac{a_{n-1}}{a_n}\right)^{n-1}. \tag{34}$$

Notice that the second inequality in (4) is equivalent to

$$\left(\frac{a_n}{a_{n+1}}\right)^r \geq \left(\frac{a_{n-1}}{a_n}\right)^r, \qquad \text{for } r > 0. \tag{35}$$

Also notice that the first inequality in (4) is equivalent to

$$\ln\left(\frac{a_{n+1}}{a_n}\right)^n \geq \ln\left(\frac{a_n}{a_{n-1}}\right)^{n-1}. \tag{36}$$

Since both sides of (35) and (36) are positive, the multiplication of (35) and (36) implies that $\frac{de_n(r)}{dr} \leq 0$ for $r > 0$. This, together with the equation $e_n(0) = 0$, implies $e_n(r) \leq 0$ for $r > 0$. Hence, (33) is true. Then so is (30). Now, it follows from (30) that

$$\frac{a_n^r}{A_n^{(r)}} < \frac{a_{n+1}^r}{A_{n+1}^{(r)}} < \cdots < \frac{a_{n+m}^r}{A_{n+m}^{(r)}},$$

implying (5). Finally, using L'Hôpital's rule, computing directly, and noticing the fact that $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing positive sequence, one gets

$$\lim_{r\to\infty}\left(\frac{A_n^{(r)}}{A_{n+m}^{(r)}}\right)^{1/r} = \frac{a_n}{a_{n+m}}, \qquad \text{for } n \geq 1.$$

Hence, (5) is sharp.

(ii) We first prove the case $r = 1$ and $m = 1$; i.e.,

$$\frac{A_n}{A_{n+1}} < \frac{G_n}{G_{n+1}}, \qquad \text{for } n \geq 1, \tag{37}$$

under condition (6), where $A_n = A_n^{(1)}$.

Write (37) as

$$\frac{nA_n}{((n+1)A_{n+1})} < \frac{nG_n}{((n+1)G_{n+1})}, \qquad \text{for } n \geq 1. \tag{38}$$

It is clear that (37) or (38) trivially holds for $n = 1$. Now we prove (38) for $n \geq 2$. By Lemma 2.1, viewing B_n as nA_n and C_n as nG_n, it suffices to show that

$$\frac{nA_n - (n-1)A_{n-1}}{(n+1)A_{n+1} - nA_n} \leq \frac{nG_n - (n-1)G_{n-1}}{(n+1)G_{n+1} - nG_n}, \qquad \text{for } n \geq 2,$$

or equivalently,

$$H_n \equiv \frac{n(G_n - G_{n-1}) + G_{n-1}}{(n+1)(G_{n+1} - G_n) + G_n} \geq \frac{a_n}{a_{n+1}}, \qquad \text{for } n \geq 2. \tag{39}$$

We proceed in our proof by using Lemmas 2.2–2.4. Given $n \geq 2$, let

$$\begin{aligned} a &= G_{n-1}, \\ b &= G_n, \\ c &= n(G_n - G_{n-1}), \\ d &= (n+1)(G_{n+1} - G_n), \\ c' &= a_n - G_{n-1}, \\ d' &= a_{n+1} - G_n. \end{aligned}$$

Suppose that $c/d \geq a/b$. Then

$$H_n = \frac{c+a}{d+b} \geq \frac{a}{b} = \frac{G_{n-1}}{G_n} \geq \frac{a_n}{a_{n+1}},$$

where the last inequality holds because of condition (6). Otherwise, we have $c/d < a/b$. Noticing that $c' > c$ by Lemma 2.2, and $c'/d' < c/d$ by Lemma 2.3, we have, by Lemma 2.4,

$$H_n = \frac{c+a}{d+b} > \frac{c'+a}{d'+b} = \frac{a_n}{a_{n+1}},$$

where the last equality holds by the definitions of a, b, c', and d'. Hence, (37) is true under condition (6).

Next, we prove the case $m = 1$; i.e.,

$$\left(\frac{A_n^{(r)}}{A_{n+1}^{(r)}}\right)^{1/r} < \frac{G_n}{G_{n+1}}. \tag{40}$$

Given $r > 0$, define a new sequence $\{S_n\}_{n=1}^{\infty}$ with $S_n = a_n^r$. Denote

$$A_n(s) = \frac{1}{n}\sum_{i=1}^{n} a_i^r, \qquad G_n(s) = \left(\prod_{i=1}^{n} a_i^r\right)^{1/n}.$$

Then

$$A_n(s) = A_n^{(r)}, \qquad G_n(s) = G_n^r. \tag{41}$$

Note that condition (6) is equivalent to

$$\frac{S_n}{S_{n+1}} \leq \frac{G_{n-1}(s)}{G_n(s)}, \qquad \text{for } n \geq 2.$$

Analogously to (37), we have, under condition (6),

$$\frac{A_n(s)}{A_{n+1}(s)} < \frac{G_n(s)}{G_{n+1}(s)}, \qquad \text{for } n \geq 1,$$

or, by (41),

$$\frac{A_n^{(r)}}{A_{n+1}^{(r)}} < \frac{G_n^r}{G_{n+1}^r}, \qquad \text{for } n \geq 1, \tag{42}$$

which implies (40).

Now, it follows from (42) that

$$\frac{A_n^{(r)}}{G_n^r} < \frac{A_{n+1}^{(r)}}{G_{n+1}^r} < \cdots < \frac{A_{n+m}^{(r)}}{G_{n+m}^r},$$

and that

$$\frac{A_n^{(r)}}{A_{n+m}^{(r)}} < \left(\frac{G_n}{G_{n+m}}\right)^r.$$

Thus, inequality (7) follows.

Finally, using L'Hôpital's rule and computing directly, we have

$$\lim_{r\to 0+} \left(\frac{A_n^{(r)}}{A_{n+m}^{(r)}}\right)^{1/r} = \frac{G_n}{G_{n+m}}, \qquad \text{for } n \geq 1.$$

Hence, (7) is sharp. The proof is complete. ∎

3. TWO COROLLARIES

We conclude our paper by presenting the following two corollaries. The first one provides a sufficient condition for (6). By using Theorem 1.2(ii), the second one solves the conjecture in a slightly generalized version. That is: k can be a nonnegative real number instead of being a nonnegative integer. Also, it shows that inequality (1) is a special case of Theorem 1.2(i).

COROLLARY 3.1. *Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing positive sequence. If*

$$\frac{a_n}{a_{n+1}} \leq \frac{A_{n-1}}{A_n}, \qquad \text{for } n \geq 2, \tag{43}$$

then

$$\frac{a_n}{a_{n+1}} < \frac{G_{n-1}}{G_n}, \qquad \text{for } n \geq 2. \tag{44}$$

PROOF. We trivially have $A_1/A_2 < G_1/G_2$. Then condition (43) implies $a_2/a_3 \leq A_1/A_2 < G_1/G_2$. So, (44) holds for $n = 2$. Suppose that (44) is valid for some $n \geq 2$. Since the first part of the proof of Theorem 1.2(ii) has been actually done by induction (Lemma 2.1) and other techniques (Lemmas 2.2–2.4), we know by (37) that $A_n/A_{n+1} < G_n/G_{n+1}$. Again, condition (43) implies $a_{n+1}/a_{n+2} \leq A_n/A_{n+1} < G_n/G_{n+1}$. The proof is complete. ∎

COROLLARY 3.2. *Let n and m be natural numbers, K a nonnegative real number. Then, we have both*

$$\frac{n+K+1}{n+m+K+1} < \frac{\left(\prod_{i=1}^{n}(i+K)\right)^{1/n}}{\left(\prod_{i=1}^{n+m}(i+K)\right)^{1/(n+m)}}. \tag{45}$$

and, for any positive real number r,

$$\begin{aligned}\frac{n+K}{n+m+K} &< \left(\frac{(1/n)\sum_{i=1}^{n}(i+K)^r}{(1/(n+m))\,(i+K)^r}\right)^{1/r}\\ &< \frac{\left(\prod_{i=1}^{n}(i+K)\right)^{1/n}}{\left(\prod_{i=1}^{n+m}(i+K)\right)^{1/(n+m)}}.\end{aligned} \tag{46}$$

PROOF. First, we note that it suffices to prove (46) for the case $m = 1$, similarly to the arguments in the last paragraph of last section. Given $K \geq 0$, for each n, define $a_n = n + k$. It is an elementary job to show that (4) holds true. Hence, the first inequality of (46) is also true by Theorem 2.1(i). Note that for $m = 1$, (45) can be read as

$$\frac{a_{n+1}}{a_{n+2}} < \frac{G_n}{G_{n+1}}, \qquad \text{for } n \geq 1. \tag{47}$$

Hence, (45) implies the second inequality of (46) by Theorem 1.2(ii). Now we only need to prove (45). Simple computation shows that $A_n/A_{n+1} = (n + 2K + 1)/(n + 2K + 2)$. It is easy to check that $a_{n+1}/a_{n+2} \leq A_n/A_{n+1}$ if and only if $K \geq 0$. Hence, (45) is true for $m = 1$ by Corollary 3.1. The proof is complete. ∎

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